

Birkhoff theorem and conformal Killing-Yano tensors

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Abstract We analyze the main geometric conditions imposed by the hypothesis of the Jebsen-Birkhoff theorem. We show that the result (existence of an additional Killing vector) does not necessarily require a three-dimensional isometry group on two-dimensional orbits but only the existence of a conformal Killing-Yano tensor. In this approach the (additional) isometry appears as the known invariant Killing vector that the \mathcal{D} -metrics admit.

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1 Introduction

The pioneering results by Jebsen [1] and Birkhoff [2] are applied to spherically symmetric vacuum solutions. Subsequently, several generalizations have been accomplished concerning either the energy content (two double eigen-values [3]), or the isometry group (plane and hyperbolical on space-like orbits [4], or maximal symmetry on time-like orbits [5]).

Thus, the two hypotheses of the original Jebsen-Birkhoff theorem have been weakened. Our aim here is to show that the present version of the former, the existence of a maximal group of symmetries on two-dimensional non-null orbits, admits a weaker statement.

In this paper we work on an oriented space-time with a metric tensor g of signature $\{-, +, +, +\}$ and metric volume element η . The Riemann, Ricci and Weyl tensors are defined as given in [6] and are denoted, respectively, by $Riem$, Ric and W . For the metric product of two vectors we write $(X, Y) = g(X, Y)$. If

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A and B are 2-tensors, $A \cdot B$ denotes the 2-tensor $(A \cdot B)^\alpha_\beta = A^\alpha_\mu B^\mu_\beta$, $A^2 = A \cdot A$, $A(X, Y) = A_{\alpha\beta} X^\alpha Y^\beta$, $A(X) = A_{\alpha\beta} X^\beta$, and $(A, B) = \frac{1}{2} A_{\alpha\beta} B^{\alpha\beta}$. For a vector X and a $(p+1)$ -tensor t , $i(X)t$ denotes the inner product, $[i(X)t]_{\underline{p}} = X^\alpha t_{\alpha\underline{p}}$, the underline denoting multi-index. And if ω is a $(p+1)$ -form, $\delta\omega$ denotes its exterior codifferential, $(\delta\omega)_{\underline{p}} = -\nabla_\alpha \omega^\alpha_{\underline{p}}$.

Note that the existence of a maximal group of symmetries on two-dimensional non-null orbits implies that the metric is conformal to a 2+2 product one with a restricted conformal factor [7], which leads to a 2+2 warped product space-time. Then, a non null Killing-Yano tensor A exists [8].

Now, we impose a weaker condition: the existence of a conformal Killing-Yano (CKY) tensor A . Then, the associated self-dual two-form $\mathcal{A} = \frac{1}{\sqrt{2}}(A - i * A)$ satisfies the CKY equation [9]:

$$3\nabla\mathcal{A} = 2i(\mathcal{Z})\mathcal{G}, \quad \mathcal{Z} \equiv \delta\mathcal{A}, \quad (1)$$

where $*$ is the Hodge dual operator. The 4-tensor \mathcal{G} is the endowed metric on the 3-dimensional complex space of the self-dual two-forms, $\mathcal{G} = \frac{1}{2}(G - i\eta)$, G being the metric on the space of 2-forms, $G_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}$. The self-dual Weyl tensor is $\mathcal{W} = \frac{1}{2}(W - i * W)$.

2 Main results

We know (see [10] and references therein) that the integrability conditions of the CKY equation (1) lead to constraints on the Petrov-Bel type. Indeed, if \mathcal{A} is a non null two-form then the space-time is type O or type D, and in type D case \mathcal{A} is an eigen-two-form of the Weyl tensor: $\mathcal{A} = e^\lambda \mathcal{U}$, \mathcal{U} being the simple unitary Weyl eigen-two-form. On the other hand, if \mathcal{A} is a null two-form then the space-time is type O or type N, and in type N case \mathcal{A} is a null eigen-two-form of the Weyl tensor: the self-dual Weyl tensor writes $\mathcal{W} = \mathcal{H} \otimes \mathcal{H}$, and $\mathcal{A} = e^\mu \mathcal{H}$.

Moreover, the integrability conditions of the CKY equation also constrains the Ricci tensor. More precisely, from the Ricci identities for the two-form \mathcal{A} and taking into account (1) we obtain:

$$2\mathcal{L}_\mathcal{Z}g = 3[\mathcal{A}, Ric], \quad (2)$$

where $\mathcal{L}_\mathcal{Z}$ denotes the Lie derivative with respect the vector field \mathcal{Z} , and for two 2-tensors P, Q , $[P, Q]$ denotes their commutator, $[P, Q] = P \cdot Q - Q \cdot P$. The commutator $[\mathcal{A}, Ric]$ vanishes if, and only if, $[\mathcal{A}, Ric] = [*A, Ric] = 0$. Then, we can state the following.

Theorem 1 *If a space-time admits a conformal Killing-Yano tensor \mathcal{A} , then $\mathcal{Z} \equiv \delta\mathcal{A}$ is a (complex) Killing vector (or it vanishes) if, and only if, $[\mathcal{A}, Ric] = 0$.*

When \mathcal{A} is a non null two-form this condition states that the Ricci tensor is of Segre types [(11)(11)] or [(1111)].

When \mathcal{A} is a null two-form this condition states that the Ricci tensor is of Segre types [(211)] or [(1111)].

Remark 1. As commented above, a CKY tensor is an eigen-two-form of the Weyl tensor. Theorem 1 states that \mathcal{Z} is a Killing vector if, and only if, the Ricci

geometry is also aligned with \mathcal{A} . Thus when $\mathcal{A} = e^\lambda \mathcal{U}$ is a non null two-form then the Ricci tensor writes $Ric = -\kappa \Pi + \Lambda g$, $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$ being the 2+2 structure tensor, and where κ and Λ are two Ricci-invariant scalars. On the other hand, when $\mathcal{A} = e^\mu \mathcal{H}$ is a null two-form then the Ricci tensor writes $Ric = \sigma \ell \otimes \ell + \Lambda g$, where ℓ is the null fundamental vector of \mathcal{H} .

Remark 2. Tachibana [11] obtained a similar equation to (2) for a n-dimensional Riemann space, and he concludes that for an Einstein space, \mathcal{Z} is a Killing vector. Theorem 1 above generalizes this result for the case of a four-dimensional space-time by considering all the compatible Ricci tensors. Thus it gives not only a necessary condition but also a sufficient one in order for \mathcal{Z} to be a Killing vector. The particular case when A is a Killing-Yano tensor has previously been considered [12].

Remark 3. Hougston and Sommers [13] proved that the \mathcal{D} -metrics (vacuum type D metrics and their charged counterpart) admit a complex Killing vector given by the divergence of a non null CKY tensor. This Killing vector is invariant since it is a Weyl concomitant. Hougston and Sommers do not quote the Tachibana paper but their result applies to a family of both vacuum and non vacuum solutions which cover the physically significant space-times for which theorem 1 applies. For short, in what follows we denote $\tilde{\mathcal{D}}$ -metrics the non conformally flat space-times admitting a non null CKY tensor \mathcal{A} such that $[\mathcal{A}, Ric] = 0$. Elsewhere [14] we have shown that the \mathcal{D} -metrics are the $\tilde{\mathcal{D}}$ -metrics with $\Lambda = \text{constant}$, and we have extended some known properties of the \mathcal{D} -metrics to the $\tilde{\mathcal{D}}$ -metrics.

The CKY equation (1) for the non null two-form $\mathcal{A} = e^\lambda \mathcal{U}$ is equivalent to the umbilical and Maxwellian character of the 2+2 structure defined by \mathcal{U} . These two properties mean that its principal directions are geodesic shear-free congruences and they determine a non null solution of the source-free Maxwell equations, and they can be written, respectively, in terms of $\{\lambda, \mathcal{U}\}$ as [9]:

$$\nabla \mathcal{U} = i(\delta \mathcal{U})[\mathcal{U} \otimes \mathcal{U} + \mathcal{G}], \quad \mathcal{U}(\delta \lambda) = d\lambda. \quad (3)$$

From this latter equation we obtain the following expression for the Killing vector:

$$\mathcal{Z} \equiv \delta \mathcal{A} = \frac{3}{2} e^\lambda \delta \mathcal{U}. \quad (4)$$

In what follows we restrict ourselves to the space-times with non constant curvature by considering either $\tilde{\mathcal{D}}$ -metrics or conformally flat metrics with a Ricci tensor $Ric = -\kappa \Pi + \Lambda g$, $\kappa \neq 0$. Under this assumption \mathcal{U} is a Riemann invariant two-form and then (3) and (4) imply:

$$\mathcal{L}_{\mathcal{Z}} \mathcal{U} = 0, \quad \mathcal{L}_{\mathcal{Z}} \bar{\lambda} = 0, \quad \mathcal{L}_{\mathcal{Z}} \bar{\mathcal{U}} = 0. \quad (5)$$

From these constraints (see [15] for a similar reasoning) we obtain the following expression for the Killing two-form associated with \mathcal{Z} :

$$d\mathcal{Z} = a\mathcal{U} + m\bar{\mathcal{U}} + \frac{4}{3} e^{-\bar{\lambda}} \bar{\mathcal{G}}(\mathcal{Z} \wedge \bar{\mathcal{Z}}) \cdot \bar{\mathcal{U}}, \quad (6)$$

where for a double two-form W and a two-form F , $W(F)$ denotes the two-form $W(F)_{\alpha\beta} = \frac{1}{2} W_{\alpha\beta\mu\nu} F^{\mu\nu}$. Then, from (4), (5) and (6) we can easily show the following.

Proposition 1 *If a non constant curvature space-time admits a non null conformal Killing-Yano tensor A such that $[A, Ric] = [*A, Ric] = 0$, then $Z_1 \equiv \delta A$ and $Z_2 \equiv \delta *A$ are Killing vectors (or they vanish) verifying: (i) The CKY tensor is Z_i -invariant: $\mathcal{L}_{Z_1} A = \mathcal{L}_{Z_2} A = 0$, (ii) If $Z_1 \wedge Z_2 \neq 0$, they define a commutative algebra: $[Z_1, Z_2] = 0$.*

Condition $Z_1 \wedge Z_2 = 0$ characterizes the Kerr-NUT solutions in the set of the \mathcal{D} -metrics [13] [14]. Their properties can be extended to the $\tilde{\mathcal{D}}$ -metrics and to the conformally flat case. Indeed, from (3), (4) and (6) (see [14] and [15] for a similar reasoning) we obtain the following.

Proposition 2 *In a non constant curvature space-time admitting a non null conformal Killing-Yano tensor A such that $[A, Ric] = [*A, Ric] = 0$, let us consider $Z_1 \equiv \delta A$ and $Z_2 \equiv \delta *A$. Then, we have the following equivalent conditions: (i) $Z_1 \wedge Z_2 = 0$, (ii) A constant duality rotation θ exists such that $F = \cos \theta A + \sin \theta *A$ is a Killing-Yano tensor, (iii) $K = F^2$ is a Killing tensor, (iv) The Killing two-form dZ of the Killing vector $Z = \delta *F$ is aligned with F : $[F, dZ] = 0$. Moreover, if these conditions hold, Z and $Y = K(Z)$ are Killing vectors (or they vanish) such that $[Z, Y] = 0$.*

Remark 4. Some of the results in propositions 1 and 2 were obtained in [13] and [16] for the case of the \mathcal{D} -metrics. In [14] we completed and partially extended these results, and here we state that all of them hold for both, the $\tilde{\mathcal{D}}$ -metrics and the conformally flat case.

In the present version of the generalized Jebsen-Birkhoff theorem the additional symmetry is defined by a hypersurface-orthogonal Killing vector. When do the Killing vectors in theorem 1 have this property? We know that a *simple* Killing-Yano tensor exists in the A-metrics and B-metrics where the Jebsen-Birkhoff theorem applies. Let us note that A is a simple (rank two) two-form if, and only if, the scalar invariant $(A, *A) = \frac{1}{2} \eta_{\alpha\beta\gamma\delta} A^{\alpha\beta} A^{\gamma\delta}$ vanishes. We show now that this condition for the CKY tensor guarantees the hypersurface-orthogonal character of the Killing vectors defined by $Z = \delta A$.

It is worth remarking that $(A, *A) = 0$ is equivalent to λ being a real scalar and, from (3), implies that the vector $\mathcal{U}(\delta\mathcal{U})$ is real, that is, the 2+2 structure is integrable (the two 2-planes are foliation) [9]. On the other hand, $Z_1 = \delta A$ and $Z_2 = \delta *A$ are hypersurface-orthogonal vectors when $I_1 = *(Z_1 \wedge dZ_1) = 0$ and $I_2 = *(Z_2 \wedge dZ_2) = 0$.

By using (4) and (6) we can compute the (real) vectors I_1 and I_2 and the codifferential of the complex vectors $\mathcal{U}(Z)$ and $\bar{\mathcal{U}}(\bar{Z})$, and we obtain:

$$\begin{aligned} I_1 + I_2 &= 2i\{m\bar{\mathcal{U}}(Z) - a\mathcal{U}(Z) + \frac{2}{3}e^{-\bar{\lambda}}[(Z, Z)\bar{\mathcal{U}}(\bar{Z}) - (Z, \bar{Z})\bar{\mathcal{U}}(Z)] - c.c.\}, \\ I_1 - I_2 &= 2i\{m\bar{\mathcal{U}}(\bar{Z}) - a\mathcal{U}(\bar{Z}) + \frac{2}{3}e^{-\bar{\lambda}}[(Z, \bar{Z})\bar{\mathcal{U}}(\bar{Z}) - (\bar{Z}, \bar{Z})\bar{\mathcal{U}}(Z)] - c.c.\}. \end{aligned} \quad (7)$$

$$\delta\mathcal{U}(Z) = a + \frac{2}{3}e^{-\lambda}(Z, Z), \quad \delta\bar{\mathcal{U}}(\bar{Z}) = m + \frac{2}{3}e^{-\bar{\lambda}}(\bar{Z}, \bar{Z}). \quad (8)$$

Let us suppose that A is a simple CKY tensor and $[A, Ric] = 0$. From theorem 1 this second condition implies that Z_1 and Z_2 are Killing vectors. Moreover, as commented above, the first one implies that λ and $\mathcal{U}(\delta\mathcal{U})$ are real, and (3) and (4) impose $\mathcal{U}(Z)$ and $\bar{\mathcal{U}}(\bar{Z})$ are real too. Then (8) imposes a and m to be real scalars.

Using all these constraints in (7) we obtain $I_1 = I_2 = 0$ and consequently Z_1 and Z_2 are hypersurface-orthogonal vectors.

Let us suppose now that Z_1, Z_2 are hypersurface-orthogonal Killing vectors. Thus $I_1 = I_2 = 0$. Moreover, from theorem 1 we have $[A, Ric] = 0$. Then Z_1 and Z_2 lie on the principal planes defined by the Ricci and/or the Weyl tensors (which are that defined by \mathcal{U}). This condition imposes $\mathcal{U}(Z)$ and $\bar{\mathcal{U}}(Z)$ to be real when $Z_1 \wedge Z_2 \neq 0$, and then from (7) and (8) we can show that λ is real and then $(A, *A) = 0$. When $Z_1 \wedge Z_2 = 0$, we can consider the Killing-Yano tensor F and $Z \equiv \delta * F$ of proposition 2, and then a similar reasoning implies $(F, *F) = 0$ if $(Z, Z) \neq 0$. Moreover $(Z, Z) = 0$ implies $\mathcal{U}(Z) = Z$, and then (8) leads to $\nabla Z = 0$, that is, the space-time is a pp-wave if $Z \neq 0$. Otherwise, when $Z = 0$, $*F$ is also a Killing-Yano tensor and the space-time is a product one. Consequently we have proven the following.

Theorem 2 *In a non constant curvature space-time admitting a non null conformal Killing-Yano tensor A such that $[A, Ric] = [*A, Ric] = 0$, let us consider $Z_1 \equiv \delta A$ and $Z_2 \equiv \delta *A$. Then, the following statements hold:*

*If $Z_1 \wedge Z_2 \neq 0$, then Z_1 and Z_2 are hypersurface-orthogonal Killing vectors if, and only if, $(A, *A) = 0$.*

*If $Z_1 \wedge Z_2 = 0$, let $F = \cos \theta A + \sin \theta *A$ be the Killing-Yano tensor given in proposition 2, and $Z \equiv \delta *F \neq 0$. Then: (i) when $(Z, Z) \neq 0$, Z is a hypersurface-orthogonal Killing vector if, and only if, $(F, *F) = 0$; (ii) when $(Z, Z) = 0$, Z is a hypersurface-orthogonal Killing vector and the space-time is a pp-wave.*

The space-time has a product metric if, and only if, $Z_1 = Z_2 = 0$.

Remark 5. When $Z_1 \wedge Z_2 \neq 0$, the hypersurface-orthogonal nature of the Killing vectors Z_1 and Z_2 is guaranteed if $(A, *A) = 0$, that is, when the scalar λ is real. Nevertheless, if the imaginary part of λ is a non vanishing constant (and then $(A, *A) \neq 0$), a constant duality rotation leads to a simple CKY tensor $A' = \cos \theta A + \sin \theta *A$, and the first statement in theorem 2 applies.

Remark 6. When a Killing tensor F exists and $Z \equiv \delta *F \neq 0$ is a null Killing vector ($(Z, Z) = 0$), then $\nabla Z = 0$ and the space-time is a pp-wave. In this case F has a constant eigenvalue and F is simple only when this eigenvalue vanishes. The only Ricci tensor of the considered type [(11)(11)] that is compatible with the integrability conditions of $\nabla Z = 0$ has a vanishing eigen-value associated with the time-like eigen-plane. Consequently, these metrics have an unclear physical meaning.

Remark 7. Theorems 1 and 2 imply the existence of one or two symmetries if the space-time has some specific geometric properties. But these symmetries do not exist when both, $Z_1 = \delta A$ and $Z_2 = \delta *A$, vanish. In this case both, A and $*A$, are Killing-Yano tensors and, as stated in the third point of theorem 2, the space-time metric is a product one. It is worth remarking that this is, precisely, the case where the generalized Jebsen-Birkhoff theorem does not apply (see [5] and [7]).

On the other hand, a simple Killing-Yano tensor A exists in the space-times where the Jebsen-Birkhoff theorem applies. Moreover, for a Killing-Yano tensor we have [17] $[A, Ric] = 0$. Then we obtain the following.

Corollary 1 *If a non constant curvature space-time admits a non null Killing-Yano tensor A and $Z \equiv \delta * A \neq 0$, then:*

*When $(Z, Z) \neq 0$, then Z is a hypersurface orthogonal Killing vector if, and only if, $[*A, Ric] = 0$ and $(A, *A) = 0$.*

*When $(Z, Z) = 0$, then Z is a hypersurface orthogonal Killing vector if, and only if, $[*A, Ric] = 0$.*

Plainly, the generalized Birkhoff theorem follows from this corollary. Ideed, a metric g admitting a maximal group of symmetries on two-dimensional non-null orbits admits the canonical form of a 2+2 warped product [7]:

$$g = v(x^0, x^1) + \phi^2(x^0, x^1)h(x^2, x^3), \quad (9)$$

where ϕ is a function, v is a Lorentzian (or Riemannian) metric and h is a Riemannian (or Lorentzian) metric of constant curvature. Then $A = \phi H$ is a simple Killing-Yano tensor for the metric (9), H being the metric volume element of the metric ϕh [8]. On the other hand, the Ricci tensor of the metric (9) has $\{x^2, x^3\}$ as an eigenplane. Consequently, a Ricci with two double eigen-values implies its alignment with $*A$. Moreover, we have $Z \equiv \delta * A = - * (d\phi \wedge H)$, which vanishes if, and only if, $d\phi = 0$. Thus, we can apply corollary 1 and we recover the generalized Birkhoff theorem.

Corollary 2 *If a space-time admits a maximal group of symmetries acting on two-dimensional non-null orbits and the Ricci tensor is of types $[(11)(11)]$ or $[1111]$ then it admits an additional hypersurface-orthogonal Killing vector provided that $d\phi \neq 0$. This Killing vector is given by $Z = - * (d\phi \wedge H)$, where H is the volume element of the group orbits.*

3 Ending comments.

Corollary generalizes the Jebsen-Birkhoff theorem because the hypothesis of a three-dimensional group of isometries on two-dimensional orbits has been weakened by considering the existence of a simple Killing-Yano tensor. Note that no symmetries are required a priori. Nevertheless, if we consider the non-conformally flat case with $\Lambda = \text{constant}$, we obtain the charged A-metrics and B-metrics, and all of them admit maximal symmetry on non null two-dimensional orbits. But these symmetries are not a hypothesis but a consequence of the field equations.

Corollary 1 not only generalizes the Jebsen-Birkhoff theorem by weakening its hypothesis, but it also offers two other new improvements. On the one hand, it is stated as a necessary and sufficient condition. On the other hand, it gives the explicit expression of the hypersurface-orthogonal Killing vector in terms of the magnitudes which appear in the hypothesis (the simple Killing tensor). This advance can be also found in the statement of the Jebsen-Birkhoff theorem given in corollary 2.

Moreover, theorem 2 is a wider generalization than corollary 1 because it establishes the existence of one or two hypersurface-orthogonal Killing vectors under weaker conditions. Besides, theorem 1 includes theorem 2 as a particular case and it shows the close relationship between the Jebsen-Birkhoff theorem and the results by Tachibana [11] and Houghshton and Sommers [13].

It is worth remarking that the different cases we found in our study correspond with extensions of the invariant classes of vacuum type D solutions presented in [15]. For each of these classes there is the charged counterpart in the set of the \mathcal{D} -metrics, and with similar invariant definitions we can consider the same classes in the $\tilde{\mathcal{D}}$ -metrics and in the conformally flat case. Thus, when $Z_1 \wedge Z_2 \neq 0$ we have C-like metrics admitting the subclass where Z_1 and Z_2 are hypersurface-orthogonal vectors (containing the strict Ehlers and Kundt C-metrics). When $Z_1 \wedge Z_2 = 0$ we have Kerr-NUT-like metrics, with a regular case characterized by $Z \wedge K(Z) \neq 0$. Note that in these three quoted cases a two-dimensional commutative group of isometries exists (see propositions 1 and 2). However, when $Z_1 \wedge Z_2 = Z \wedge K(Z) = 0$ we obtain A-NUT-like and B-NUT-like metrics and only a real Killing vector Z is defined by the CKY tensor A . When Z is a hypersurface-orthogonal vector we have the subfamily where corollary 1 applies.

Theorem 1 also holds when A is a null CKY tensor. Nevertheless, the other results presented herein only apply when A is a non null CKY tensor, and their possible generalization to the null case is not evident. This study and its relationship with previously known Birkhoff-like results for null orbits [5] [18] will be considered elsewhere.

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